Adomian Decomposition and Modified Variational Iteration Methods for Linear Fractional Integro-Differential Equations

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Abstract

In this paper, we present a comparative study between the modified variational iteration method and Adomian decomposition method. The study outlines the significant features of the two methods, for solving linear fractional integro-differential equations. From the computational viewpoint, the variational iteration method is more efficient, convenient and easy to use.

Keywords: Fractional integro-differential equations, Adomian decomposition method (ADM), Modified variational iteration method (MVIM), Caputo fractional derivative.

1. Introduction

In recent years various analytical and numerical methods have been applied for the approximate solutions of fractional integro-differential equations. Fractional models have been shown by many scientists to adequately describe the operation of variety of physical and biological processes and systems ([4-7]). Consequently, considerable attention has been given to the solution of fractional integral equations of physical interest. Since most fractional integro-differential equations do not have exact analytic solutions, approximation and numerical techniques, therefore, are used extensively. Recently, the Adomian decomposition method ([2],[12]) and Variational iteration method ([10],[11],[13],[14]) have been used for solving a wide range of problems.

To construct an approximate solution to linear integro-differential equations of fractional order. The variational iteration method was first proposed by He ([10],[11]) and has been worked out over a number of years by many authors. This method has been shown to effectively, easily and accurately solve a large class of linear problems. Generally, one or two iterations lead to high accurate solutions.


Recently Adomian decomposition method is extended for multiterm diffusion-wave equations of fractional order(Gejji and (2008))also Mittal et al(2008) solved fractional integro-differential equations by Adomian decomposition method. The method works very well and solutions of nonlinear problems can be obtained very easily and accurately. The method is well reviewed in (Adomian (1988,1992,1994)). The method involving splitting an equation into linear and nonlinear parts and assume that it has a infinite series solution. This series has to be truncated for practical purpose but by adding more terms it is possible to...
get arbitrary close to the exact solution in a specific domain.
In this paper, we are concerned with the numerical solution of following linear fractional integro-differential equations:

\[ D^\alpha \Psi(x) = g(x) + \int_0^x k(x,t)\Psi(t)dt, \quad 0 \leq x, t \leq 1, \quad (1.1) \]

with the following supplementary conditions:

\[ \Psi^{(i)}(0) = \delta_i, \quad n - 1 < \alpha \leq n, \quad n \in \mathbb{N}, \quad (1.2) \]

where \( D^\alpha \Psi(x) \) indicates \( \alpha \) Caputo fractional derivative of \( \Psi(x) \), \( g(x) \) and \( k(x,t) \) are given functions, \( x \) and \( t \) are real variables varying in the interval \([0, 1]\), and \( \Psi(x) \) is the unknown function to be determined.

2. Basic definitions

In this section some basic definitions and properties of fractional calculus theory which are necessary for the formulation of the problem are given.

**Definition 2.1:** A real function \( f(x), x > 0 \), is said to be in the space \( C_\mu, \mu \in \mathbb{R} \), if there exists a real number \( p > \mu \) such that \( f(x) = x^p f_1(x) \), where \( f_1(x) \in C[0, 1] \).

**Definition 2.2:** A function \( f(x), x > 0 \), is said to be in the space \( C_m^\mu, m \in \mathbb{N} \cup \{0\} \), if \( f^{(m)} \in C_\mu \).

**Definition 2.3:** The left sided Riemann-Liouville fractional integral operator of order \( \alpha \geq 0 \) of a function \( f \in C_\mu, \mu \geq -1 \), is defined as [15]:

\[ J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad \alpha > 0, \quad x > 0, \quad (2.1) \]

\[ J^0 f(x) = f(x). \]

**Definition 2.4:** Let \( f \in C_{m+1}^m, m \in \mathbb{N} \cup \{0\} \). Then the Caputo fractional derivative of \( f(x) \) is defined as [4-7]

\[ D^\alpha f(x) = \begin{cases} J^{m-\alpha} f^m(x), & m - 1 < \alpha \leq m, \quad m \in \mathbb{N}, \\ \frac{D^m f(x)}{Dx^m}, & \alpha = m. \end{cases} \quad (2.2) \]

Hence, we have the following properties:

1. \( J^\alpha J^{\nu} f = J^{\alpha + \nu} f, \quad \alpha, \nu > 0, \quad f \in C_\mu, \quad \mu > 0. \quad (2.3) \)
2. \( J^\alpha x^{\gamma} = \frac{\Gamma(\gamma + 1)}{\Gamma(\alpha + \gamma + 1)} x^{\alpha + \gamma}, \quad \alpha > 0, \quad \gamma > -1, \quad x > 0. \quad (2.4) \)
3. \( J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0, \quad m - 1 < \alpha \leq m. \quad (2.5) \)
4. \( D^\alpha J^\alpha f(x) = f(x), \quad x > 0, \quad m - 1 < \alpha \leq m. \quad (2.6) \)
5. \( D^\alpha C = 0, \quad C \text{ is a constant.} \quad (2.7) \)
6. \( D^\alpha x^\beta = \begin{cases} 0, & \beta \in \mathbb{N}_0, \ \beta < [\alpha], \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, & \beta \in \mathbb{N}_0, \ \beta \geq [\alpha]. \end{cases} \)

where \([\alpha]\) denoted the smallest integer greater than or equal to \(\alpha\) and \(\mathbb{N}_0 = \{0, 1, 2, \ldots\}\. \\

3. Analysis of The Modified Variational Iteration Method

In this section, for the convenience of the reader, we first present a brief review of He’s variational iteration method [16]. Then we will propose the reliable modification of the VIM [1] for solving fractional integro-differential equations. Here, we consider the following fractional functional equation

\[
Lu + Ru + Nu = g(x), 
\]

where \(L\) is the fractional order derivative, \(R\) is a linear differential operator, \(N\) represents the nonlinear terms, and \(g\) is the source term. By using (2.5) and applying the inverse operator \(L^{-1}_x\) to both sides of (3.1), and using the given conditions, we obtain

\[
u = f - L^{-1}_x[Ru] - L^{-1}[Nu] \tag{3.2}
\]

where the function \(f\) represents the terms arising from integrating the source term \(g\) and from using the given conditions, all are assumed to be prescribed. 

The basic character of He’s method is the construction of a correction functional for (3.1), which reads

\[
u_{n+1}(x) = \nu_n(x) + \int_0^x \lambda(s)[Lu_n(s) + Ru_n(s) + Nu_n(s) - g(s)]ds, \tag{3.3}
\]

where \(\lambda\) is a Lagrange multiplier which can be identified optimally via variational theory [9], \(\nu_n\) is the \(n\)th approximate solution, and \(\tilde{\nu}_n\) denotes a restricted variation, i.e., \(\delta\tilde{\nu}_n = 0\). 

To solve (3.1) by He’s VIM, we first determine the Lagrange multiplier \(\lambda\) that will be identified optimally via integration by parts. Then the successive approximations \(u_n(x), \ n \geq 0\), of the solution \(u(x)\) will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function \(u_0\). The approximation \(u_0\) may be selected by any function that just satisfies at least the initial and boundary conditions. With determined \(\lambda\), then several approximations \(u_n(x), \ n \geq 0\), follow immediately. Consequently, the exact solution may be obtained by using

\[
\lim_{n \to 1} u_n(x) = u(x). \tag{3.4}
\]

In summary, we have the following variational iteration formula for (3.2)

\[
\begin{align*}
u_0(x) & \text{ is an arbitrary initial guess,} \\
u_{n+1}(x) = & \nu_n(x) + \int_0^x \lambda(s)[Lu_n(s) + Ru_n(s) + Nu_n(s) - g(s)]ds, 
\end{align*} \tag{3.5}
\]

or equivalently, for (3.2), according to [16]:

\[
\begin{align*}
u_0(x) & \text{ is an arbitrary initial guess,} \\
u_{n+1}(x) = & f(x) - L^{-1}_x[Ru] - L^{-1}[Nu], 
\end{align*} \tag{3.6}
\]

where the multiplier Lagrange \(\lambda\) has been identified.

It is important to note that He’s VIM suggests that the \(u_0\) usually defined by a suitable trial-function with some unknown parameters or any other function that satisfies at least the initial and boundary conditions. This assumption made by He ([8],[9]) and others will be slightly varied, as will be seen in the discussion.

The MVIM, that was introduced by Ghorbani et al [1], can be established based on the assumption
that the function $f(x)$ of the iterative relation (3.6) can be divided into two parts, namely $f_0(x)$ and $f_1(x)$. Under this assumption, we set

$$f(x) = f_0(x) + f_1(x).$$  \quad (3.7)

According to the assumption, (3.7), and by the relationship (3.6), we construct the following variational iteration formula

\begin{align*}
\begin{cases}
  u_0(x) = f_0(x), \\
  u_1(x) = f(x) - L_x^{-1}[Rf_0(x)] - L_x^{-1}[Nf_0(x)], \\
  u_{n+1}(x) = f(x) - L_x^{-1}[Ru_n(x)] - L_x^{-1}[Nu_n(x)],
\end{cases}
\end{align*}

where the multiplier Lagrange $\lambda$, has been identified. Here, a proper selection was proposed for the components $u_0(x)$ and $u_1(x)$. The suggestion was that only the part $u_0$ be assigned to the zeroth component i.e $u_0$. An important observation that can be made here is that the success of the proposed method depends mainly on the proper choice of the functions $f_0$ and $f_1$. As will be seen from the examples below, this selection of $u_0$ will result in a reduction of the computational work and accelerate the convergence. Furthermore, this proper selection of the components $u_0$ and $u_1$ may provide the solution by using one iteration only. To give a clear overview of the content of this study. We have chosen several fractional integro-differential equations with the nonlocal boundary conditions.

4. Analysis of The Adomian Decomposition Method

Consider the equation (1.1) with initial condition (1.2) where $D^\alpha$ is the operator defined as (2.2). Operating with $J^\alpha$ on both sides of the equation (1.1) as follows:

$$\Psi(x) = \sum_{k=0}^{m-1} \psi^{(k)}(0^+) \frac{x^k}{k!} + J^\alpha g(x) + J^\alpha \int_0^x k(x,t)\Psi(t)dt, \quad 0 \leq x,t \leq 1, \quad (4.1)$$

Adomian decomposition method defines the solution by the series:

$$\Psi(x) = \sum_{i=0}^{\infty} \psi_i(x), \quad (4.2)$$

where the components $\psi_0(x), \psi_1(x), \psi_2(x), ..., \psi_i(x), ...$ are determined recursively by

$$\psi_0(x) = \sum_{k=0}^{m-1} \psi^{(k)}(0^+) \frac{x^k}{k!} + J^\alpha g(x),$$

$$\psi_{i+1}(x) = J^\alpha g(x) + J^\alpha \int_0^x k(x,t)\psi_i(t)dt, \quad (4.3)$$

decomposition method suggests that $0^{th}$ component $\psi_0(x)$ be defined by the initial conditions and the function $g(x)$ as described above. The other components namely $\psi_0(x), \psi_1(x), ...$ etc. are derived recursively.

5. Numerical examples

In this section, some numerical examples of linear fractional integro-differential equations are presented to illustrate the above results. All results are obtained by using Maple 16.

Example 5.1

Consider the following fractional integro-differential equation:

$$D^{\frac{3}{2}}\Psi(x) = \frac{2401}{276} x^\frac{3}{2} - \frac{1}{5} x^5 \sin(x) - x \sin(x) + \int_0^x \sin(x)\Psi(t)dt, \quad (5.1)$$
subject to $\Psi(0) = 1$ with the exact solution $\Psi(x) = 1 + x^4$.

By assuming $L = D^2$ and applying the inverse operator $L_x^{-1}$ to both sides of (5.1), we have

$$\Psi(x) = 1 + J^2 \left[ \frac{2401}{276} \frac{x^{\frac{22}{5}}}{\Gamma\left(\frac{22}{5}\right)} \right] - J^2 \left[ \frac{1}{5} x^5 \sin(x) + x \sin(x) \right] + J^2 \int_0^x \sin(x) \Psi(t) dt. \quad (5.2)$$

According to the original VIM (3.5), and corresponding the recursive scheme (3.6) and by using (3.7), we obtain

$$g(x) = g_0(x) + g_1(x) = 1 + J^2 \left[ \frac{2401}{276} \frac{x^{\frac{22}{5}}}{\Gamma\left(\frac{22}{5}\right)} \right] - J^2 \left[ \frac{1}{5} x^5 \sin(x) + x \sin(x) \right],$$

by assuming

$$g_0(x) = 1 + x^4, \quad g_1(x) = -J^2 \left[ \frac{1}{5} x^5 \sin(x) + x \sin(x) \right], \quad \text{and with starting of the initial approximation } \Psi_0(x) = g_0(x) = 1 + x^4, \text{ we have}$$

$$\Psi_0(x) = 1 + x^4,$$

$$\Psi_1(x) = 1 + x^4 - J^2 \left[ \frac{1}{5} x^5 \sin(x) + x \sin(x) \right] + L_x^{-1} [\Psi_0(x)],$$

$$\Psi_1(x) = 1 + x^4 - J^2 \left[ \frac{1}{5} x^5 \sin(x) + x \sin(x) \right] + J^2 \left[ \frac{1}{5} x^5 \sin(x) + x \sin(x) \right] = 1 + x^4,$$ \quad (5.4)

$$\Psi_{n+1}(x) = 1 + x^4 - J^2 \left[ \frac{1}{5} x^5 \sin(x) + x \sin(x) \right] + L_x^{-1} [\Psi_n(x)] \Rightarrow \Psi_{n+1}(x) = 1 + x^4, \quad n \geq 0.$$

In view of (5.4), we obtain the approximate solution $\Psi(x) = 1 + x^4$, which is the same of the exact solution.

Now applying Adomian decomposition method for equation (5.2) by using equation (4.2) we get:

$$\sum_{i=0}^{\infty} \Psi_i(x) = 1 + J^2 \left[ \frac{2401}{276} \frac{x^{\frac{22}{5}}}{\Gamma\left(\frac{22}{5}\right)} \right] - J^2 \left[ \frac{1}{5} x^5 \sin(x) + x \sin(x) \right] + J^2 \int_0^x \sin(x) \sum_{i=0}^{\infty} \Psi_i(t) dt, \quad (5.5)$$

by using equation (5.5) we obtain the as follows:

$$\Psi_0(x) = 1 + J^2 g(x),$$

$$\Psi_0(x) = 1 + x^4 - 0.07160086639 x^{\frac{22}{5}} + 0.007058904187 x^{\frac{22}{5}} - 0.003033879182 x^{\frac{22}{5}} + \ldots,$$

$$\Psi_1(x) = J^2 \lambda \int_0^x k(x, t) \Psi_0(t) dt,$$

$$\Psi_1(x) = 0.05264769589 x^{\frac{22}{5}} - 0.007058904186 x^{\frac{22}{5}} + 0.003033879182 x^{\frac{22}{5}} + \ldots,$$

$$\Psi_2(x) = J^2 \lambda \int_0^x k(x, t) \Psi_1(t) dt,$$

$$\Psi_2(x) = -5.773754204 \times 10^{-12} x^{\frac{22}{5}} + 1.089352272 \times 10^{-13} x^{\frac{22}{5}} - 1.315207672 \times 10^{-15} x^{\frac{22}{5}} + \ldots,$$

$$\Psi_3(x) = J^2 \lambda \int_0^x k(x, t) \Psi_2(t) dt,$$

$$\Psi_3(x) = -2.129464558 \times 10^{-10} x^{\frac{22}{5}} + 1.152235978 \times 10^{-11} x^{\frac{22}{5}} - 3.718717854 \times 10^{-24} x^{\frac{22}{5}} + \ldots,$$

hence $\sum_{i=0}^{\infty} \Psi_i(x) = \Psi_0 + \Psi_1 + \Psi_2 + \Psi_3 + \ldots$.

The approximation solution for $\alpha = 5/7$ is

$$\Psi(x) = 1 + x^4 - 2.129464558 \times 10^{-10} x^{\frac{22}{5}} + 1.152235978 \times 10^{-11} x^{\frac{22}{5}} - 3.718717854 \times 10^{-24} x^{\frac{22}{5}} - \ldots.$$
Table 1 and figure 1 show the comparison between Modified variational method and Adomian Decomposition method, we find that Modified variational method is better than Adomian decomposition method.

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Figure 1: Numerical results of Example 5.1.
Example 5.2
Consider the following fractional integro-differential equation:

\[ D^\frac{3}{4} \Psi(x) = \frac{3}{4} x^\frac{3}{4} \sqrt{\frac{3}{\pi}} \Gamma\left(\frac{3}{4}\right)(3x - 2) - \frac{1}{4} x^4 \exp(x) - \frac{1}{3} x^3 \exp(x) + \int_0^x \exp(x) t \Psi(t) dt, \tag{5.6} \]

subject to \( \Psi(0) = 0 \) with the exact solution \( \Psi(x) = x^2 - x \).

By assuming \( L = D^\frac{3}{4} \) and applying the inverse operator \( L^{-1} \) to both sides of (5.6), we have

\[ \Psi(x) = 0 + J^\frac{3}{4} \left[ \frac{3}{4} x^\frac{3}{4} \sqrt{\frac{3}{\pi}} \Gamma\left(\frac{3}{4}\right)(3x - 2) \right] - J^\frac{3}{4} \left[ \frac{1}{4} x^4 \exp(x) + \frac{1}{3} x^3 \exp(x) \right] + J^\frac{3}{4} \int_0^x \exp(x) t \Psi(t) dt, \tag{5.7} \]

According to the original VIM (3.5), and corresponding the recursive scheme (3.6) and by using (3.7), we obtain

\[ g(x) = g_0(x) + g_1(x) = 0 + J^\frac{3}{4} \left[ \frac{3}{4} x^\frac{3}{4} \sqrt{\frac{3}{\pi}} \Gamma\left(\frac{3}{4}\right)(3x - 2) \right] - J^\frac{3}{4} \left[ \frac{1}{4} x^4 \exp(x) + \frac{1}{3} x^3 \exp(x) \right], \]

by assuming

\[ g_0(x) = x^2 - x, \quad g_1(x) = -J^\frac{3}{4} \left[ \frac{1}{4} x^4 \exp(x) + \frac{1}{3} x^3 \exp(x) \right], \tag{5.8} \]

and with starting of the initial approximation \( \Psi_0(x) = g_0(x) = x^2 - x \), we have

\[ \Psi_0(x) = x^2 - x, \]

\[ \Psi_1(x) = x^2 - x - J^\frac{3}{4} \left[ \frac{1}{4} x^4 \exp(x) + \frac{1}{3} x^3 \exp(x) \right] + L^{-1}_x[\Psi_0(x)], \tag{5.9} \]

\[ \Psi_{n+1}(x) = x^2 - x - J^\frac{3}{4} \left[ \frac{1}{4} x^4 \exp(x) + \frac{1}{3} x^3 \exp(x) \right] + L^{-1}_x[\Psi_n(x)] \Rightarrow \Psi_{n+1}(x) = x^2 - x, \quad n \geq 0. \]

In view of (5.9), we obtain the approximate solution \( \Psi(x) = x^2 - x \), which is the same of the exact solution.

Now applying Adomian decomposition method for equation (5.7) by using equation (4.2), we get:

\[ \sum_{i=0}^{\infty} \Psi_i(x) = 0 + J^\frac{3}{4} \left[ \frac{3}{4} x^\frac{3}{4} \sqrt{\frac{3}{\pi}} \Gamma\left(\frac{3}{4}\right)(3x - 2) \right] - J^\frac{3}{4} \left[ \frac{1}{4} x^4 \exp(x) + \frac{1}{3} x^3 \exp(x) \right] + J^\frac{3}{4} \int_0^x \exp(x) t \sum_{i=0}^{\infty} \Psi_i(t) dt, \tag{5.10} \]

by using equation (5.10) we obtain as the follows:

\[ \Psi_0(x) = J^\frac{3}{4} g(x), \]

\[ \Psi_0(x) = x^2 - x + 0.02913191738x^{14/3} - 0.02570463301x^{17/3} - 0.019278474x^{20/3} - \ldots, \]

\[ \Psi_1(x) = J^\frac{3}{4} \int_0^x k(x,t) \Psi_0(t) dt, \]

\[ \Psi_1(x) = 0.02570463301x^{17/3} + 0.019278474x^{20/3} + 0.01056125138x^{23/3} - 0.02913191740x^{14/3} + \ldots, \]

\[ \Psi_2(x) = J^\frac{3}{4} \int_0^x k(x,t) \Psi_1(t) dt, \]

\[ \Psi_2(x) = -0.006885695970x^{19/3} - 0.007399887556x^{22/3} - 0.003123602081x^{25/3} + 0.000542358779x^{34/3} + \ldots, \]

\[ \Psi_3(x) = J^\frac{3}{4} \int_0^x k(x,t) \Psi_2(t) dt, \]

\[ \Psi_3(x) = 4.873687408 \times 10^{-8} x^{19} + 0.000007245575142x^{16} + 0.0000141363675x^{15} - 0.0003449563982x^{10} + \ldots, \]
hence $\sum_{i=0}^{\infty} \Psi_i(x) = \Psi_0 + \Psi_1 + \Psi_2 + \Psi_3 + \ldots$.

The approximation solution for $\alpha = 2/3$ is

$$\Psi(x) = x^2 - x - 2 \times 10^{-11}x^{14/3} - 1 \times 10^{-11}x^{17/3} - 6 \times 10^{-13}x^{28/3} - 1 \times 10^{-13}x^{37/3} - 5 \times 10^{-13}x^{31/3} + \ldots.$$ 

Table 2 and figure 2 show the comparison between Modified variational method and Adomian Decomposition method, we find that Modified variational method is better than Adomian decomposition method.
Example 5.3
Consider the following fractional integro-differential equation:

\[ D^\frac{\alpha}{2} \Psi(x) = \frac{125}{14} x^\frac{\alpha}{2} \sin\left(\frac{\pi}{2} x\right) \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\pi} - \frac{1}{3} x^\frac{\alpha}{2} + \int_0^x \sqrt{t} \Psi(t) dt, \]  
(5.11)

subject to \( \Psi(0) = 0 \) with the exact solution \( \Psi(x) = 2x^3 \).

By assuming \( L = D^\frac{\alpha}{2} \) and applying the inverse operator \( L_x^{-1} \) to both sides of (5.11), we have

\[ \Psi(x) = 0 + J^\frac{\alpha}{2} \left[ \frac{125}{14} x^\frac{\alpha}{2} \sin\left(\frac{\pi}{2} x\right) \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\pi} \right] - J^\frac{\alpha}{2} \frac{1}{3} x^\frac{\alpha}{2} + J^\frac{\alpha}{2} \int_0^x \sqrt{t} \Psi(t) dt, \]  
(5.12)

According to the original VIM (3.5), and corresponding the recursive scheme (3.6) and by using (3.7), we obtain

\[ g(x) = g_0(x) + g_1(x) = 0 + J^\frac{\alpha}{2} \left[ \frac{125}{14} x^\frac{\alpha}{2} \sin\left(\frac{\pi}{2} x\right) \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\pi} \right] - J^\frac{\alpha}{2} \frac{1}{3} x^\frac{\alpha}{2} \]

by assuming

\[ g_0(x) = 2x^3, \quad g_1(x) = -J^\frac{\alpha}{2} \frac{1}{3} x^\frac{\alpha}{2}, \]  
(5.13)

and with starting of the initial approximation \( \Psi_0(x) = g_0(x) = 2x^3 \), we have

\[ \Psi_1(x) = 2x^3 - J^\frac{\alpha}{2} \frac{1}{3} x^\frac{\alpha}{2} + L_x^{-1} [\Psi_0(x)], \]  
(5.14)

\[ \Psi_2(x) = 2x^3 - J^\frac{\alpha}{2} \frac{1}{3} x^\frac{\alpha}{2} + J^\frac{\alpha}{2} \frac{1}{3} x^\frac{\alpha}{2} = 2x^3, \]

\[ \Psi_{n+1}(x) = 2x^3 - J^\frac{\alpha}{2} \frac{1}{3} x^\frac{\alpha}{2} + L_x^{-1} [\Psi_n(x)] \Rightarrow \Psi_{n+1}(x) = 2x^3, \quad n \geq 0. \]

In view of (5.14), we obtain the approximate solution \( \Psi(x) = 2x^3 \), which is the same of the exact solution.

Now applying Adomian decomposition method for equation (5.12) by using equation (4.2), we get:

\[ \sum_{i=0}^\infty \Psi_i(x) = 0 + J^\frac{\alpha}{2} \left[ \frac{125}{14} x^\frac{\alpha}{2} \sin\left(\frac{\pi}{2} x\right) \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\pi} \right] - J^\frac{\alpha}{2} \frac{1}{3} x^\frac{\alpha}{2} + J^\frac{\alpha}{2} \int_0^x \sqrt{t} \sum_{i=0}^\infty \Psi_i(t) dt, \]  
(5.15)

by using equation (5.15) we obtain as the follows:

\[ \Psi_0(x) = J^\frac{\alpha}{2} g(x), \]
\[ \Psi_0(x) = 2x^3 - 0.1011008935x^{71/10}, \]
\[ \Psi_1(x) = J^\frac{\alpha}{2} \int_0^x k(x,t) \Psi_0(t) dt, \]
\[ \Psi_1(x) = -0.002324006305x^{56/5} + 0.1011008935x^{71/10}, \]
\[ \Psi_2(x) = J^\frac{\alpha}{2} \int_0^x k(x,t) \Psi_1(t) dt, \]
\[ \Psi_2(x) = -0.00003160247042x^{153/10} + 0.002324006305x^{56/5}, \]
\[ \Psi_3(x) = J^\frac{\alpha}{2} \int_0^x k(x,t) \Psi_2(t) dt, \]
\[ \Psi_3(x) = -2.896664654 \times 10^{-7}x^{97/5} + 0.00003160247042x^{153/10}, \]
\[ \text{hance } \sum_{i=0}^\infty \Psi_i(x) = \Psi_0 + \Psi_1 + \Psi_2 + \Psi_3 + \ldots. \]
The approximation solution for $\alpha = 3/5$ is

$$\Psi(x) = 2x^3 - 2.896664654 \times 10^{-7} x^{97/5}.$$ 

Table 3 and figure 3 show the comparison between Modified variational method and Adomian Decomposition method, we find that Modified variational method is better than Adomian decomposition method.

<table>
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<tr>
<th>$x$</th>
<th>Exact</th>
<th>Approx variational</th>
<th>Approx adomian</th>
<th>Error variational</th>
<th>Error adomian</th>
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<td>0.002</td>
<td>0.002</td>
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<td>0.016</td>
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<td>0.054</td>
<td>0.054</td>
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<td>0.128</td>
<td>0.128</td>
<td>0</td>
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</tr>
<tr>
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<td>0.250</td>
<td>0.250</td>
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<td>4.187128641 $\times$ 10 $^{-13}$</td>
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<td>2.000</td>
<td>1.999999710</td>
<td>0</td>
<td>2.896664654 $\times$ 10 $^{-7}$</td>
</tr>
</tbody>
</table>

Figure 3: Numerical results of Example 5.3.
**Conclusion**

In this paper we study the numerical solution of three examples by using Modified variational iteration method and Adomian decomposition method which derive a good approximation. The study shows that the techniques require less computational work than existing approaches while supplying quantitatively reliable results.

**References**


