Time and Frequency Response Analysis of Fuzzy Controller

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Abstract

In this paper the control system is designed using Proportional-Integral-Differential (PID) controller and Fuzzy controller with model transfer function of the plant in which temperature is to be controlled and Time response and Frequency Response analysis is done both theoretically and by simulation. The model is designed using MatLab Simulink and response is testing for the same and a comparative result is finally presented.

Keywords: Transfer Function, Time response, Frequency response, FIS, Member Function, Rule Base, Rules, Surface, Simulation

1. Introduction

In this, the system is represented using block diagram using Simulink. Then the mathematical model for Plant is derived. The time response analysis is carried out to calculate response parameters, frequency response and Stability Analysis is carried out theoretically and by simulation.

2. Plant Model

The plant considered is having Heating process and a Thermometer system.

2.1 Heating Process

It consists of heating element; we also take in consideration the mass of tank $m$, heat transfer coefficient $H_0$, total area $\alpha$ and specific heat of the tank material $C$.

Figure 1. Block Diagram of System
The transfer function of the heating process can be represented as \[4\]

\[
T(s) = \frac{k}{s\tau + 1}
\]  

(1)

where \( k = \frac{1}{aH_0} \) \& \( \tau = \frac{mC}{aH_0} \) and \( \tau \) is time constant.

\subsection*{2.2 Temperature Sensor}

In this we are considering a thermometer as sensor and its mathematical model is having a resistance connected in series with a capacitor and voltage across it changes with temperature.

\[\text{Figure 3. Electrical Analog of Thermometer}\]

Then transfer function can be calculated by first applying Kirchoff’s Law and finding \( e_i \) and \( e_o \)

\[ Ri + \frac{1}{C} \int idt = e_i \]  

(2)

\[ \frac{1}{C} \int idt = e_o \]  

(3)

Then by Laplace Transform of Eqn. 2 & Eqn.3

\[ RI(s) + \frac{1}{C} \frac{I(s)}{s} = Ei(s) \]

\[ I(s) = Eo(s) \]

Now as \( Eo(s)/Ei(s) \) represents the transfer function of Thermometer as \[1\],[2] 

\[ \frac{Eo(s)}{Ei(s)} = \frac{1}{RCs + 1} \]  

(4)

Now as per the setup we are having the a single Plant so the combined transfer function can be represented as

\[
\frac{k}{(s\tau + 1)(RCs + 1)}
\]

(5)

From experimental data \( RC=1 \) and \( \tau=0.2s \) the transfer function is

\[
\frac{1}{s(0.4s^2 + 1.2s + 2)}
\]

(6)

\[
\frac{1}{s^3 + 3s^2 + 5}
\]

(7)

\subsection*{2.3 PID Controller}

As modern control system is automated control system and most of the applications it is desirable to have auto – control instead of manual control. It is a feedback type control system with combination of Proportional, Integral & Derivative control over the system.

\[\text{Figure 4. PID Controller}\]

The proportional control has an important factor as proportional gain \( K_p \), it amplifies the output proportional to the ‘error’ signal received. In integral control the ‘error’ is accumulated and multiplied by integral gain \( K_i \). In derivative control the output changes at the rate of change of ‘error’ with respect to time multiplied by derivative gain \( K_d \).

In case of combined PID control the relation is represented as \[1\],[2],[6]

\[
u(t) = K_p e(t) + K_i \int_0^t e(t)dt + K_d \frac{\partial e(t)}{\partial t}
\]

(8)

\subsection*{2.4 Fuzzy Controller}

Fuzzy controller is designed using Fuzzy Logic Toolbox. \[8\] First Fuzzy Inference System (FIS) Editor is used with two inputs ‘Error’ and ‘DErr’ and one output. After performing multiple ranges of value of ‘Error’, dError & Output finally it was observed that desired performance is available for the ranges,
Error = -20 to 20, DError = -2 to 2 and output = -5 to 5. The FIS type is Mamdani with Implication & Aggregation as min and Defuzzification method used is centroid method.

Input one is ‘Error’ with triangular membership functions with five states NB, NS, Z, PS, PB which are the linguistic variables for Negative Big, Negative Small, Zero, Positive Small & Positive Big. Ranging from -20 to 20

Another input is ‘DErr’ too with triangular membership functions with five states NB, NS, Z, PS, PB which are the linguistic variables for Negative Big, Negative Small, Zero, Positive Small & Positive Big. Ranging from -2 to 2

Output is member function with triangular membership functions with five states NB, NS, Z, PS, PB which are the linguistic variables for Negative Big, Negative Small, Zero or Set value, Positive Small & Positive Big. Ranging from -5 to 5

Then according to the transient response available for PID controller Fuzzy rule base is designed to get desired improvement in control output.

Lastly the 3D plot of the Rule implementation is observed using surface viewer.

3. Time Response Analysis
Time response comprises of transient response and steady state response. Transient response is which goes from initial to final state and steady state response is refers to behavior of output of system from $t \to \infty$ [1]

### 3.1 Transient Response

A transient response is the out characteristic of a system, it generally exhibits damped oscillations before reaching steady state. In a transient response following parameters are considered.[1][2]

a.) Delay Time ($t_d$): It is the time required for the response to reach half the final value for first time.

b.) Rise Time ($t_r$): It is the time required by the response to reach to its final or 90% value from 0 to 10%.

c.) Peak Time ($t_p$): It is the time required by the response to reach to first peak of the overshoot.

d.) Maximum Overshoot ($M_p$): It is the maximum peak value reached reference to the unity value.

e.) Settling Time ($t_s$): It is the time required by the response to reach to become stable within an allowable range.

#### 3.1.a PID Controller

Let us consider unit step response for PID with Transfer function represented by Eqn.7

Then we will perform transient analysis [6]

From the Fig.11 first of all we will find the damping ratio $\zeta$ and natural frequency $\omega_n$. We find that:

$M_{p1} = 1.58$  $M_{p2} = 1.06$

Then we will find the transfer function associated with the value of $\zeta$ and $\omega_n$

$$ C(s) = \frac{0.27123}{s^2 + 0.35914s + 0.27123} \quad (9) $$

Output response for the above transfer function is

#### 1) RiseTime ($t_r$) = $\frac{\pi - \phi}{\omega_d}$

$$ \phi = \tan^{-1} \sqrt{1 - \zeta^2} \quad \& \quad \omega_d = \omega_n \sqrt{1 - \zeta^2} \quad (8) $$
2) \( \text{Peak Time}(t_p) = \frac{\pi}{\omega_d} \)

3) \( \text{Maximum overshoot}(M_p) = e^{-\zeta \sqrt{1-\zeta^2}} \)

4) \( \text{Settling Time}(t_s) = \frac{1}{\zeta \omega_n} \)

\[ \phi = \tan^{-1} \frac{\sqrt{1-(0.3448)^2}}{0.3448} = 1.218 \text{ rad} \]

\[ \omega_d = 0.5208 \sqrt{1-(0.3448)^2} = 0.4888 \text{ rad/sec} \]

1) \( t_r = \frac{3.14 - 1.218}{0.4888} = 3.93 \text{ sec} \)

2) \( t_p = \frac{3.14}{0.4888} = 6.427 \text{ sec} \)

3) \( M_p = e^{-0.3448 \cdot 3.14} / \sqrt{1-(0.3448)^2} = 0.3617 \)

4) \( t_s = \frac{4}{0.3448 \times 0.5208} = 22.27 \)

According to the theoretical calculation the transient response parameters are as above and according to simulation are:

\[ \text{rise time} = 3.95, \text{ peak time} = 6.45 \]

\[ \text{max overshoot} = 0.3154 \]

\[ \text{settling time} = 21 \]

3.1. b Steady State response

Now as can observe that control system is Type-I and the steady state error for unit step input for Type-I is equal to zero[1][2] so \( e_s = 0 \).

3.1. c Fuzzy Controller

Let us consider unit step response for Fuzzy with Transfer function represented by Eqn.7

Then we will perform transient analysis [6]

From the Fig.13 first of all we will find the damping ratio \( \zeta \) and natural frequency \( \omega_n \).

\[ \zeta = \frac{1}{\sqrt{1 + \left(\frac{2\pi}{\ln d}\right)^2}} \]

\[ \omega_n = \frac{2\pi}{\left(3.05 \sqrt{1-(0.3728)^2}\right)} \]

Then we find the transfer function associated with the value of \( \zeta \) and \( \omega_n \).
\[
\frac{C(s)}{R(s)} = \frac{4.9284}{s^2 + 1.6552s + 4.9284}
\]

(10)

1) \(\text{RiseTime}(t_r) = \frac{\pi - \phi}{\omega_d}\)
\[
\phi = \tan^{-1} \sqrt{1 - \zeta^2} \quad \text{&} \quad \omega_d = \omega_n \sqrt{1 - \zeta^2}
\]

2) \(\text{PeakTime}(t_p) = \frac{\pi}{\omega_d}\)
3) \(\text{Maximum overshoot}(M_p) = e^{-\phi/\sqrt{1-\zeta^2}}\)
4) \(\text{SettlingTime}(t_s) = \frac{1}{\zeta \omega_n}\)
\[
\phi = \tan^{-1} \sqrt{\frac{1 - (0.3728)^2}{0.3728}} = 1.1887 \text{rad}
\]
\[
\omega_d = 2.220 \sqrt{1 - (0.3728)^2} = 2.059 \text{ rad/sec}
\]
1) \(t_r = \frac{3.14 - 1.1887}{2.059} = 0.9484 \text{sec}\)
2) \(t_p = \frac{3.14}{2.059} = 1.525 \text{sec}\)
3) \(M_p = e^{-0.3728 \times 3.14} / \sqrt{1 - (0.3728)^2} = 0.3373\)
4) \(t_s = \frac{4}{0.3728 \times 2.220} = 4.833\)

According to the theoretical calculation the transient response parameters are as above and according to simulation are
rise_time = 0.95, peak_time = 1.55
max_overshoot = 0.2826
settling_time = 4.8

4. Frequency Response Analysis

Frequency response analysis refers to response of the system to input sine function.[1][2] We will use two methods for representing Frequency response analysis, first is 'Bode Plot' which is same as step response in time domain and other is 'Nyquist Plot' which will through light on absolute and relative stability analysis.

4.1 Bode Plot Analysis for PID

It is graphical representation of system transfer function in frequency area. It has two plots magnitude plot and phase plot. For drawing first we get the open loop transfer function in frequency domain by replacing \(s\) by \(j\omega\). So we have \(G(j\omega)\) from Eqn. 9 as
\[
G(j\omega) = \frac{0.27123}{(j\omega)^2 + 0.35914 j\omega}
\]
(11)
1. Magnitude of resonance point \(M_r\). It is maximum value dimension in \(G(j\omega)\). It is related to dimensions of overshoot and damping ratio
\[
\zeta, M_r = \frac{1}{2 \zeta \sqrt{1 - \zeta^2}}
\]
\[
M_r = \frac{1}{2 \times 0.3448 \sqrt{1 - (0.3448)^2}} = 1.54
\]
It also expresses relative stability if \(1.0 < M_r < 1.4\) i.e 0dB - 3dB, it corresponds to effective damping ratio \(0.4 < \zeta < 0.7\) for satisfactory performance. If the value is greater than 1.5 then multiple overshoot will be available. [2]. But in this case \(\zeta < 0.4\).
2. Resonant Frequency \(\omega_r\) it is the frequency at maximum resonant point and is related to natural frequency \(\omega_n\) as
\[
\omega_r = \omega_n \sqrt{1 - 2\zeta^2}
\]
Indicates the speed of response.
\[
\omega_r = 0.5208 \sqrt{1 - 2(0.3448)^2} = 0.454
\]
3. Bandwidth \(Bw\) It indicates how the system traces the sine wave. Greater the bandwidth better the high frequency passes, it is related to rise time and in general is proportional to response time. It is recommended that \(M_r\) should be small and \(\omega_r\) should be high.[2]
\[
Bw = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + \zeta^4}}
\]
\[
x = \sqrt{2 - 4(0.3448)^2 + (0.3448)^4}
\]
\[
Bw = 0.5208 \sqrt{1 - 2(0.3448)^2 + x}
\]
\[
x = 0.7370
\]
4. Gain & Phase Margin \((Gm & Pm)\) Gain margin is difference between gain magnitude and 0dB at phase transit frequency \(\omega_p\). Phase margin is difference between the phase and \(-180^\circ\) when gain is 0dB at gain transit frequency \(\omega_g\)
\[
Pm = 100 \times \zeta \Rightarrow 100 \times 0.3448 = 34.48
If the gain margin is negative below 0dB then it indicates stability. Greater the better. These margins indicate the margin gain and phase has for system stability. From simulation the values of $M_r=1.5449$, $\omega_r=0.4642$, $Bw=0.7055$, $Gm=\infty$, $Pm=37.80$, $\omega_p=\infty$ & $\omega_g=0.4629$ are.

4.2. Bode Plot Analysis for Fuzzy

It is graphical representation of system transfer function in frequency area. It has two plots magnitude plot and phase plot. For drawing first we get the open loop transfer function in frequency domain by replacing $s$ by $j\omega$. So we have $G(j\omega)$ from Eqn. 10 as $G(j\omega) = \frac{4.9284}{(j\omega)^2 + 1.6552j\omega}$ (12)

1. Magnitude of resonance point ($M_r$). It is maximum value dimension in $G(j\omega)$. It is related to dimensions of overshoot and damping ratio

$$\zeta M_r = \frac{1}{2\zeta \sqrt{1-\zeta^2}}$$

$$M_r = \frac{1}{2 \times 0.3728 \sqrt{1-(0.3728)^2}} = 1.445$$

It also expresses relative stability if $1.0 < M_r < 1.4$ i.e 0dB – 3dB, it corresponds to effective damping ratio $0.4<\zeta<0.7$ for satisfactory performance. If the value is greater than 1.5 then multiple overshoot will be available. [2]. But in this case $\zeta<0.4$.

2. Resonant Frequency ($\omega_r$) it is the frequency at maximum resonant point and is related to natural frequency $\omega_n$ as $\omega_r = \omega_n \sqrt{1-2\zeta^2}$ it indicates the speed of response.

$$\omega_r = 2.220\sqrt{1-2(0.3728)^2} = 1.886$$

3. Bandwidth ($Bw$) It indicates how the system traces the sine wave. Greater the bandwidth better the high frequency passes, it is related to rise time and in general is proportional to response time. It is recommended that $M_r$ should be small and $\omega_r$ should be high.[2] $Bw = \omega_r \sqrt{1-2\zeta^2 + \sqrt{2-4\zeta^2 + \zeta^4}}$

$$x = \sqrt{2-4(0.3728)^2 + (0.3728)^4}$$

$$Bw = 2.220\sqrt{1-2(0.3728)^2 + x} = 3.085$$

4. Gain & Phase Margin ($Gm$ & $Pm$) Gain margin is difference between gain magnitude and 0dB at phase transit frequency $\omega_p$. Phase margin is difference between the phase and -180˚ when gain is 0dB at gain transit frequency $\omega_g$.

$Pm = 100 \times \zeta \Rightarrow 100 \times 0.3728 = 37.28$

If the gain margin is negative below 0dB then it indicates stability. Greater the better. These margins indicate the margin gain and phase has for system stability.

From simulation the values of $M_r=1.445$, $\omega_r=1.873$, $Bw=3.05$, $Gm=\infty$, $Pm=40.53$, $\omega_p=\infty$ & $\omega_g=1.93$ are.

4.3 Stability Analysis by Nyquist Plot for PID

Nyquist stability criterion determines the stability of closed loop system from its open loop frequency response. It has two conditions for stability [1][2][4][5]
i) If the nyquist contour for open loop system does not encircle \(-1+j0\) point then the close loop system is stable.

ii) Along with the i) another condition is, there should be no poles to right of the \(s\)-plane

Let us consider the open loop transfer function represented by Eqn.11. The Nyquist plot for the same is as

\[ G_m = \left( \frac{KT_1T_2}{T_1 + T_2} \right)^{-1} \]

\[ P_m = \tan^{-1} \left[ 2\zeta \sqrt{\frac{1}{\sqrt{(4\zeta^4 + 1) - 2\zeta^2}}} \right] \]

In case when \(G_m = \infty\) then \(P_m\) is the correct measure of relative stability.

So we find \(P_m\) by putting the value of damping factor in the formula

\[ P_m = \tan^{-1} \left( 2 \times 0.3448 \frac{1}{\sqrt{x - 2 \times (0.3448)^2}} \right) \]

\[ x = \sqrt{(4 \times (0.3448)^4 + 1) = 1.027} \]

\[ P_m = 54.19^\circ \]

From the Fig.17 we can calculate \(P_m\) as

\[ P_m = \tan^{-1} \left( \frac{0.809}{0.5878} \right) = 53.99^\circ \]

This value of \(P_m\) varies with value of \(K\) gain.

### 4.4 Stability Analysis by Nyquist Plot for Fuzzy

Nyquist stability criterion determines the stability of closed loop system from its open loop frequency response. It has two conditions for stability [1][2][4][5]

iii) If the nyquist contour for open loop system does not encircle \(-1+j0\) point then the close loop system is stable.

iv) Along with the i) another condition is, there should be no poles to right of the \(s\)-plane

Let us consider the open loop transfer function represented by Eqn.12. The Nyquist plot for the same is as

The gain margin \(G_m\) and phase margin \(P_m\) is calculated as

**Figure.16 Nyquist plot**

The Nyquist stability criterion can also be expressed as \(Z= N+P\) where

\(Z\) is number of zeros in right- half plane of \(s\)-plane.

\(N\) is number of encircles of \(-1+j0\)

\(P\) is number of Poles in right- half plane of \(s\)-plane.

As it is clear from the Fig.14 that \(Z=0, P=0 & N=0\). So we can conclude that the system is stable.

Relative stability can also be analyzed; we observe that as the plot moves towards \(-1+j0\) it tends towards instability. The relative stability can be measured in terms of \(a & \phi\). They refer to Gain margin and Phase margin. The value \(a\) refers to the point when the locus crosses the axis at 180° in this \(a= -0.809\) and the angle \(\phi\) refers to the angle the locus forms with the axis. [2]
N is number of encircles of -1+j0
P is number of Poles in right-half plane of s-plane.
As it is clear from the Fig.18 that Z=0, P=0 & N=0.
So we can conclude that the system is stable.
Relative stability can also be analyzed; we observe
that as the plot moves towards -1+j0 it tends
towards instability. The relative stability can be
measured in terms of a & \( \phi \). They refer to Gain
margin and Phase margin. The value a refers to the
point when the locus crosses the axis at 180˚ in this
a= -0.755 and the angle \( \phi \) refers to the angle the
locus forms with the axis. [2]

The gain margin \( G_m \) and phase margin \( P_m \) is
calculated as

\[
G_m = \left( \frac{KT_1}{T_2} \right)^{-1}
\]

\[
P_m = \tan^{-1}[2\zeta\sqrt{(4\zeta^2 + 1) - 2\zeta^2}]
\]

In case when \( G_m = \infty \) then \( P_m \) is the correct measure
of relative stability.
So we find \( P_m \) by putting the value of damping factor
in the formula

\[
P_m = \tan^{-1}[2 \times 0.3728\sqrt{\frac{1}{\sqrt{x} - 2 \times (0.3728)^2}}]
\]

\[
x = \sqrt{(4 \times (0.3728)^4 + 1} = 1.037
\]
\[
P_m = 40.53^\circ
\]
From the Fig.19 we can calculate \( P_m \) as

\[
P_m = \tan^{-1} \frac{0.755}{0.624} = 49.6^\circ
\]

This value of \( P_m \) varies with value of \( K \) gain.

5. Conclusion

In this paper Time response and Frequency response
analysis is carried out theoretically and by simulation
for temperature control system using PID and Fuzzy
controller and the different parameters are calculated.
It is observed that both the systems are under damped
and Fuzzy controller performance is acceptably
enhanced compared to PID as rise time and settling
time is less signifying better speed, overshoot is also
less indicating less oscillations in output. Larger
bandwidth, Phase margin indicate higher frequency
operation ease, although gain margin is infinity buy
still stability be a constraint.

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